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A NEW CONCEPT OF CONVERGENCE SPACE

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This paper is dedicated to my colleagues Jirí Adamék and Walter Tholen on the occasion of their sixtieth birthdays

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Abstract: The notion of b-convergence is presented for studying preuniform convergence spaces in the sense of Preuß (1993) and set-convergence spaces introduced by Wyler in 1989 from a common point of view.

The well-known supertopologies as defined by Doitchinov in 1964 and also the filtermerotopies in the sense of Katétov (1965) can be integrated as well. Even the grill-defined presupernear operators, introduced by the author (1999) are contained in this new broader concept.

Moreover, we discuss all the properties for describing categories in the realm of Convenient Topology, especially the properties of being cartesian closed or extensional.

1. Introduction

In this paper we present a new type of convergence, which generalizes the "classical" ones of set-convergence in the sense of Wyler [11] and of preuniform convergence in the sense of Preuß [9] by bringing them both together.

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Thus, a comprehensive theory of convergence space is being established, which enables us to simultaneously express generalized "topological" and "uniform" aspects.

Hence the branches of Convenient Topology and of non-symmetric Convenient Topology are both involved and can be discussed in connection with *generalized* Cauchy spaces or filter spaces, respectively [6].

As a basic concept we consider uniform filters converging to bounded subsets, thus defining by suitable axioms the so-called b-convergences. Morphisms between the corresponding spaces are then defined in an obvious way, i.e., they are bounded maps which preserve uniform filters and so-called b-continuous functions. The resulting category ***b-CONV*** is "topological" in the sense that it is *fibre-small*, *initially complete* and moreover has the *terminal separator property*. So in general, subspaces and products, or quotients and sums as well are simultaneously formed by supplying the corresponding sets with the initial (respectively final) b-convergence with respect to the given data (see e.g., [9]). Moreover, we show that *pointed* b-convergence leads us to a *strong* topological universe in which the constructs ***TOP*** and ***UNIF*** can both be embedded in particularly nice fashion.

2. Basic concepts

As usual, PX denotes the power set of a set X, and we use $\mathcal{B}^X \subseteq$ $\subseteq PX$ to denote a collection of *bounded* subsets of X, also known as B-sets. Moreover, $\boldsymbol{FIL}(X \times X)$ denotes the set of all *uniform* filters on X.

2.1 Definition. We call a pair $(\mathcal{B}^X, \tau)$ consisting of a B-set $\mathcal{B}^X$ and a function $\tau : \mathcal{B}^X \longrightarrow P(\boldsymbol{FIL}(X \times X))$ a *b-convergence space* and τ a *b-convergence* (on $\mathcal{B}^X$), if the following axioms are satisfied:
(bC1) $B' \subseteq B \in \mathcal{B}^X$ implies $B' \in \mathcal{B}^X$;
(bC2) $\emptyset \in \mathcal{B}^X$;
(bC3) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(bC4) $x \in X$ implies $\dot{x} \times \dot{x} \in \tau(\{x\})$;
(bC5) $\tau(\emptyset) = \{P(X \times X)\}$;
(bC6) $B \in \mathcal{B}^X$, $\mathfrak{U} \in \tau(B)$ and $\mathfrak{U} \subseteq \mathfrak{V} \in \boldsymbol{FIL}(X \times X)$ imply $\mathfrak{V} \in \tau(B)$.
(Here $\dot{x}$ denotes the filter generated by the set $\{x\}$.) In general, for filters $\mathfrak{F}$ and $\mathfrak{G}$ their *cross product* is defined by

$$\mathfrak{F} \times \mathfrak{G} := \{\, T \subseteq X \mid \exists F \in \mathfrak{F}\, \exists G \in \mathfrak{G}.\ T \supseteq F \times G \,\}.$$

If $\mathfrak{U} \in \tau(B)$ for some $B \in \mathcal{B}^X$, we say the uniform filter $\mathfrak{U}$ *b-converges* to B.

Given two b-convergence spaces $(\mathcal{B}^X, \tau_X)$ and $(\mathcal{B}^Y, \tau_Y)$, a function $f : X \longrightarrow Y$ is called *b-continuous* iff it is *bounded*, which means
(c1) $\{ f[B] \mid B \in \mathcal{B}^X \} \subseteq \mathcal{B}^Y$,
and in addition we have that f *preserves uniform filters* in the sense that
(c2) $B \in \mathcal{B}^X$ and $\mathfrak{U} \in \tau_X(B)$ imply $(f \times f)(\mathfrak{U}) \in \tau_Y(f[B])$, where

$$(f \times f)(\mathfrak{U}) := \{ V \subseteq Y \times Y \mid (f \times f)^{-1}[V] \in \mathfrak{U} \}.$$

Moreover, we denote the corresponding category by ***b-CONV***, and mention here its interesting property of being *topological* (see Th. 4.1).

2.2 Examples. (i) Consider a set-convergence space $(X, \mathcal{M}^X, q)$, where X is a set, $\mathcal{M}^X$ is a B-set $q \subseteq \boldsymbol{FIL}(X) \times \mathcal{M}^X$ relates filters on X with bounded sets subject to the following conditions (O. Wyler):
(SC1) $\dot{A}\, q\, A$ for any $A \in \mathcal{M}^X$, where $\dot{A} := \{B \subseteq X \mid B \supseteq A\}$;
(SC2) $\mathcal{F} \in \boldsymbol{FIL}(X)$ implies $\mathcal{F}\, q\, \emptyset$ iff $\mathcal{F} = PX$;
(SC3) $A \in \mathcal{M}^X$, $\mathcal{F}_1\, q\, A$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \in \boldsymbol{FIL}(X)$ imply $\mathcal{F}_2\, q\, A$.
These data induce a function τ_q from $\mathcal{M}^X$ into $P(\boldsymbol{FIL}(X \times X))$ by setting for each $A \in \mathcal{M}^X$

$$\tau_q(A) := \{\mathfrak{U} \in \boldsymbol{FIL}(X \times X) \mid \exists \mathcal{F} \in \boldsymbol{FIL}(X).\ \mathcal{F}\, q\, A \text{ and } \dot{A} \times \mathcal{F} \subseteq \mathfrak{U}\}.$$

(ii) A special case arises for a surrounding system (neighborhood system) $(\mathcal{M}^X, \Theta)$ on a set X, where $\mathcal{M}^X$ is a B-set and $\Theta : \mathcal{M}^X \longrightarrow \boldsymbol{FIL}(X)$ is a function into the set of all filters on X (including the zero-filter PX) satisfying the following properties [3]:
(SS1) $\Theta(\emptyset) = PX$;
(SS2) $A \in \mathcal{M}^X$ and $U \in \Theta(A)$ imply $U \supseteq A$.
Then in analogy we may set for each $A \in \mathcal{M}^X$:

$$\tau_\Theta(A) := \{ \mathfrak{U} \in \boldsymbol{FIL}(X) \mid \dot{A} \times \Theta(A) \subseteq \mathfrak{U} \}.$$

(iii) Let (X, J_X) be a preuniform convergence space, where $J_X \subseteq \boldsymbol{FIL}(X \times X)$ satisfies the following two conditions (Preuß):
(PUC1) $x \in X$ implies $\dot{x} \times \dot{x} \in J_X$;
(PUC2) $\mathfrak{U}_1 \in J_X$ and $\mathfrak{U}_1 \subseteq \mathfrak{U}_2 \in \boldsymbol{FIL}(X \times X)$ imply $\mathfrak{U}_2 \in J_X$.
In this case we consider PX as a B-set and define a function from PX into $P(\boldsymbol{FIL}(X \times X))$ by setting $\tau_X(B) := J_X$ for each nonempty $B \subseteq X$, and $\tau_X(\emptyset) := \{P(X \times X)\}$.

(iv) At last, consider a presupernear space $(\mathcal{B}^X, N)$, where $\mathcal{B}^X$ is a B-set (on a set X) and $N : \mathcal{B}^X \longrightarrow P(P(P(X)))$ is a function satisfying the following conditions [7]:

(SN1) $\mathcal{N}_1 \ll \mathcal{N}_2 \in N(B)$ implies $\mathcal{N}_1 \in N(B)$;
(SN2) $N(\emptyset) = \{\emptyset\}$ and $\mathcal{B}^X \notin N(B)$ for each $B \in \mathcal{B}^X$;
(SN3) $x \in X$ implies $\{\{x\}\} \in N(\{x\})$.
Moreover, let $(\mathcal{B}^X, N)$ be *grill-defined*, which means that in addition we have
(G) for each $\mathcal{N} \in N(B)$ there exists a grill $\mathcal{G} \in \boldsymbol{GRL}(X)$ with $\mathcal{N} \subseteq \mathcal{G} \in$ $\in N(B)$.
Recall that $\mathcal{G} \subseteq PX$ is called a *grill* on the set X (G. Choquet), provided that
(G1) $\emptyset \notin \mathcal{G}$;
(G2) $G_1 \cup G_2 \in \mathcal{G}$ iff $G_1 \in \mathcal{G}$ or $G_2 \in \mathcal{G}$.
Then we set for each $B \in \mathcal{B}^X$:

$$\tau_N(B) := \big\{\mathfrak{U} \in \boldsymbol{FIL}(X \times X) \mid \exists \mathcal{G} \in \boldsymbol{GRL}(X). \\ \mathcal{G} \in N(B) \text{ and } \sec \mathcal{G} \times \sec \mathcal{G} \subseteq \mathfrak{U}\big\}.$$

Recall that $\sec \mathcal{G} := \{\, T \subseteq X \mid \forall G \in \mathcal{G}.\, G \cap T \neq \emptyset \,\}$.

3. Categorical concepts

Sets of bounded subsets of a set X are described axiomatically by the postulates (bc1) through (bc3) (see also Def. 2.1).

Having B-sets $\mathcal{B}^X$ and $\mathcal{B}^Y$, respectively, a function $f : X \longrightarrow Y$ is called *bounded*, if it preserves bounded sets (see again Def. 2.1).

The category $\boldsymbol{BOUND}$ with pairs $(X, \mathcal{B}^X)$ consisting of a set X and a corresponding B-set $\mathcal{B}^X$ as objects and bounded maps as morphisms is a topological universe, which means it is cartesian closed and extensional and hence has universal one-point extensions. If no confusion is possible, we consider the B-sets $\mathcal{B}^X$ as objects of $\boldsymbol{BOUND}$.

We recall the defining conditions for a concrete category $\boldsymbol{C}$ to be called *topological*:
(CT1) "Existence of initial structures": For any set X, any family $(X_i, T_i)_I$ of $\boldsymbol{C}$-objects indexed by a class I, and any family $(f_i : X \longrightarrow X_i)_I$ of maps indexed by I, there exists a unique $\boldsymbol{C}$-structure T on X that is *initial* with respect to $(X, f_i, (X_i, T_i), I)$. I.e., for any C-object (Y, S) a function $g : Y \longrightarrow X$ is a C-morphism from (Y, S) to (X, T) iff for every $i \in I$ the composite map $f_i \circ g : Y \longrightarrow X_i$ is a C-morphism from (Y, S) to (X_i, T_i).
(CT2) "Fibre-smallness": For any set X the $\boldsymbol{C}$-fibre, i.e., the class of all $\boldsymbol{C}$-structures on X, is a set.

(CT3) "Terminal separator property": For any set X with cardinality 1 there exists precisely one $\boldsymbol{C}$-structure.
Moreover, a topological category (construct) is *cartesian closed* (i.e., has natural function space structures), provided that for any pair (A, B) of $\boldsymbol{C}$-objects the set $\boldsymbol{Mor}(A, B)$ of all $\boldsymbol{C}$-morphisms from A to B can be equipped with the structure of a $\boldsymbol{C}$-object, denoted by $\boldsymbol{Pow}(A, B)$ and called *power-object* or *natural function space*, such that the following are satisfied:
(1) The evaluation map $e : A \times \boldsymbol{Pow}(A, B) \longrightarrow B$ defined by $e(a, f) :=$ $:= f(a)$ for each pair $(a, f) \in A \times \boldsymbol{Pow}(A, B)$ is a $\boldsymbol{C}$-morphism;
(2) for each $\boldsymbol{C}$-object C and each $\boldsymbol{C}$-morphism $f : A \times C \longrightarrow B$ the map $\hat{f} : C \longrightarrow \boldsymbol{Pow}(A, B)$ defined by $\hat{f}(a)(c) := f(a, c)$ is a $\boldsymbol{C}$-morphism.
For a topological category $\boldsymbol{C}$ with *universal one-point extensions* the last expression means that every $\boldsymbol{C}$-object A can be embedded via the addition of a single point ∞ into an object $A^* := A \cup \{\infty\}$ such that the following holds:
- For every $\boldsymbol{C}$-morphism $f : U \longrightarrow A$ from a subspace U of a $\boldsymbol{C}$-object B into A the unique function $f^* : B \longrightarrow A^*$ defined by

$$f^*(b) := \begin{cases} f(b), & \text{if } b \in U, \\ \infty & \text{otherwise} \end{cases}$$

  is a $\boldsymbol{C}$-morphism.

For basic literature concerning these definitions the reader is referred to the book of Preuß [9].

4. Convenient properties in the realm of $\boldsymbol{b\text{-}CONV}$

The aim of Convenient Topology (see [9]) consists in the study of "strong topological universes", in which "convergence" structures are available. Furthermore, such a strong topological universe should be easily described by means of suitable axioms and should not be too large.

Thus, the construct $\boldsymbol{PUCONV}$ of preuniform convergence spaces in the sense of Preuss is a good candidate for this purpose in Convenient Topology.

As already pointed out there also exist convergence structures (e.g., set-convergences, supertopologies and grill-defined presupernear operators), which cannot be subsumed by the above-mentioned construct $\boldsymbol{PUCONV}$. This motivated our broader concept of b-convergence.

Now we will examine whether ***b-CONV*** or some of its interesting subcategories satisfies the proposed axioms for being a topological universe, or strong topological universe, respectively.

4.1 Theorem. ***b-CONV*** *is a topological category.*

Proof. For a B-set $B \in \mathcal{B}^X$ and a class I let $(\mathcal{B}^{X_i}, \tau_i)_I$ be a family of b-convergence spaces and $(f_i : X \longrightarrow X_i)_I$ a family of bounded maps from $\mathcal{B}^X$ to $\mathcal{B}^{X_i}$. We set

$$\tau_{\text{in}}(B) = \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{\mathfrak{U} \in \boldsymbol{FIL}(X \times X) \mid \forall i \in I.\, (f_i \times f_i)(\mathfrak{U}_i) \in \tau_i(f_i[B])\}, & \text{if } B \neq \emptyset. \end{cases}$$

Then τ_{in} is the initial b-convergence on $\mathcal{B}^X$ with respect to the given data.

To (bc4): For $x \in X$ we have $(f_i \times f_i)(\dot{x} \times \dot{x}) = \{(f_i(x), f_i(x))\}^{\cdot} \in \in \tau_i(\{f_i(x)\})$. All the remaining axioms are easy to verify. By definition, the functions f_i are b-continuous. Now let $(\mathcal{B}^Y, \Gamma)$ be a b-convergence space and $g : Y \longrightarrow X$ be a map such that $f_i \circ g$ is b-continuous from $(\mathcal{B}^Y, \Gamma)$ to $(\mathcal{B}^{X_i}, \tau_i)$ for every $i \in I$. Consider $\mathfrak{U} \in \Gamma(B)$ and $B \in \mathcal{B}^Y \setminus \{\emptyset\}$. Then we have $(f_i \times f_i)((g \times g)(\mathfrak{U})) = ((f_i \circ g) \times (f_i \circ g)(\mathfrak{U})) \in \tau_i(f_i[B])$ for each $i \in I$. Hence, $(g \times g)(\mathfrak{U}) \in \tau_{\text{in}}(g[B])$ follows, which shows that f is b-continuous from $(\mathcal{B}^Y, \Gamma)$ to $(\mathcal{B}^X, \tau_{\text{in}})$.

The other two axioms of being a topological category are obviously satisfied. $\Diamond$

4.2 Definition. A b-convergence τ on $\mathcal{B}^X$ and the corresponding pair $(\mathcal{B}^X, \tau)$ are called *isoform*, provided

(if) $\emptyset \neq B_2 \subseteq B_1 \in \mathcal{B}^X$ implies $\tau(B_2) \subseteq \tau(B_1)$.

4.3 Remark. Isotone set-convergence spaces $(X, \mathcal{M}^X, q)$, where q satisfies in addition

(SC4) $A_2 \subseteq A_1 \in \mathcal{M}^X$ and $\mathcal{F}\, q\, A_2$ imply $\mathcal{F}\, q\, A_1$,

lead us to isoform b-convergences τ_q. Grill-defined pseudosupernear operators N determine isoform b-convergences τ_N as well, where N satisfies axioms (SN1) through (SN4).

ib-CONV denotes the full subcategory of ***b-CONV*** spanned by the isoform b-convergences.

4.4 Remark. Cartesian closedness of topological constructs implies that quotient maps are finitely productive, but not necessarily productive (i.e., not closed under the construction of arbitrary products). We will call a topological construct satisfying this latter property *strong*.

4.5 Lemma. *Let X be a set, $(\mathcal{B}^{X_i}, \tau_i)_I$ be a family of isoform b-convergence spaces and $(f_i : X_i \longrightarrow X)_I$ be a family of maps. Then we set*

$$\tau(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \big\{\mathfrak{U} \in \boldsymbol{FIL}(X \times X) \mid \exists i \in I \exists \mathfrak{U}_i \in \\ \in \tau_i(f_i^{-1}[B]).\, (f_i \times f_i)(\mathfrak{U}_i) \subseteq \mathfrak{U}\big\} \cup \big\{\dot{x} \times \dot{x} \mid x \in X\big\}, & \text{if } B \neq \emptyset. \end{cases}$$

*Consequently τ_X is the final **ib-CONV**-structure on $\mathcal{B}^X$ with respect to the given data.*

Proof. This is evident. $\Diamond$

4.6 Remark. If $(f_i : X_i \longrightarrow X)_I$ is an epi-sink in ***Set*** (i.e., $X = \bigcup\big\{f_i[X_i] \mid i \in I\big\}$), then for every $B \in \mathcal{B}^X \setminus \{\emptyset\}$ we have

$$\tau_{\text{fin}}(B) = \big\{\mathfrak{U} \in \boldsymbol{FIL}(X \times X) \mid \exists i \in I\, \exists \mathfrak{U}_i \in \tau_i(f_i^{-1}[B]).\, (f_i \times f_i)(\mathfrak{U}_i) \subseteq \mathfrak{U}\big\}.$$

4.7 Theorem. ***ib-CONV*** *is a strong topological construct.*

Proof. Let $((\mathcal{B}^{X_i}, \tau_{X_i}) \xrightarrow{f_i} (\mathcal{B}^{Y_i}, \tau_{Y_i}))_I$ be a non-empty family of quotient maps in ***ib-CONV*** indexed by a set I, and consider the corresponding product diagram in ***ib-CONV***

$$\begin{array}{ccc} (\mathcal{B}^X, \tau_X) & \xrightarrow{\prod f_i} & (\mathcal{B}^Y, \tau_Y) \\ {\scriptstyle p_{X_i}}\big\downarrow & & \big\downarrow{\scriptstyle p_{Y_i}} \\ (\mathcal{B}^{X_i}, \tau_{X_i}) & \xrightarrow[f_i]{} & (\mathcal{B}^{Y_i}, \tau_{Y_i}) \end{array},$$

where $(\mathcal{B}^X, \tau_X) := \prod_{i \in I}(\mathcal{B}^{X_i}, \tau_{X_i})$ and $(\mathcal{B}^Y, \tau_Y) := \prod_{i \in I}(\mathcal{B}^{Y_i}, \tau_{Y_i})$.

Since all f_i are surjective, $\prod f_i$ is surjective as well. For each $i \in I$, and every $B_i \in \mathcal{B}^{Y_i} \setminus \{\emptyset\}$ we have

$$\tau_{Y_i}(B_i) := \{\, \mathfrak{U} \in \boldsymbol{FIL}(Y_i \times Y_i) \mid \exists \mathfrak{U}_i \in \tau_{X_i}(f_i^{-1}[B_i]).\, (f_i \times f_i)(\mathfrak{U}) \subseteq \mathfrak{U}\,\}$$

because f_i is a quotient map. For every $B \in \mathcal{B}^Y \setminus \{\emptyset\}$ define

$$\tau'_Y(B) := \{\mathfrak{V} \in \boldsymbol{FIL}(Y \times Y) \mid \exists \mathfrak{W} \in \tau_X((\textstyle\prod f_i)^{-1}[B]).(\prod f_i \times \prod f_i)(\mathfrak{W}) \subseteq \mathfrak{V}\}$$

which implies

$$\tau'_Y(B) = \tau_Y(B) := \{\mathfrak{U} \in \boldsymbol{FIL}(Y \times Y) \mid (p_{Y_i} \times p_{Y_i})(\mathfrak{U}) \in \tau_{Y_i}(p_i[B])\}.$$

This means that $\prod f_i$ is a quotient map. $\Diamond$

4.8 Theorem. ***ib-CONV*** *is extensional.*

Proof. For an isoform b-convergence space $(\mathcal{B}^X, \tau)$ we put $X^* := X \cup \cup\, \{\infty\}$ and $\mathcal{B}^* := \mathcal{B}^X \cup \{\{\infty\}\}$, and we define a b-convergence τ^* on $\mathcal{B}^*$ by setting:

$$\tau^*(B) = \begin{cases} \{P(X^* \times X^*)\}, & \text{if } B = \emptyset; \\ \{\dot{\infty} \times \dot{\infty}\}, & \text{if } B = \{\infty\}; \\ \{\mathfrak{U}^* \in \boldsymbol{FIL}(X^* \times X^*) \mid \exists \mathfrak{U} \in \tau(B).\, \mathfrak{U}^* \supseteq \bar{\mathfrak{U}}\}, & \text{if } B \in \mathcal{B}^X \setminus \{\emptyset\} \end{cases}$$

with $\bar{\mathfrak{U}} := \{\, \bar{U} \mid U \in \mathfrak{U} \,\}$ and $\bar{U} := U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$.

To (bc4): $x \in X^*$ implies $x = \infty$ or $x \in X$. In the first case $\dot{x} \times \dot{x} = \dot{\infty} \times \dot{\infty} \in \tau^*(\{\infty\}) = \tau^*(\{x\})$. In the second case choose $\dot{x} \times \dot{x} \in \in \tau(\{x\})$, hence $\overline{\dot{x} \times \dot{x}} \subseteq \dot{x} \times \dot{x}$ follows, which shows $\dot{x} \times \dot{x} \in \tau^*(\{x\})$.

All the other axioms for being an isoform b-convergence are trivially satisfied.

$(\mathcal{B}^X, \tau)$ is a subspace of $(\mathcal{B}^*, \tau^*)$, because $\mathcal{B}^X = \{\, B \cap X \mid B \in \mathcal{B}^* \,\}$ and τ is initial with respect to the inclusion $i : X \longrightarrow X^*$. The latter follows since each $\mathfrak{U} \in \tau(B)$ is the trace of $\overline{\mathfrak{U}}$ with respect to X. Hence we obtain $\tau^*/\mathcal{B}^X = \tau$.

Now, let $(\mathcal{B}^Y, \tau_Y)$ be an isoform b-convergence space, and $f : (\mathcal{B}^Z, \tau_Z) \longrightarrow (\mathcal{B}^X, \tau_X)$ be a b-continuous function from a subspace $(\mathcal{B}^Z, \tau_Z)$ of $(\mathcal{B}^Y, \tau_Y)$. We have to show that $f^* : (\mathcal{B}^Y, \tau_Y) \longrightarrow (\mathcal{B}^*, \tau^*)$ is again b-continuous. Consider, without restriction, $B \in \mathcal{B}^Y \setminus \{\emptyset\}$ and $\mathfrak{U} \in \tau_Y(B)$, hence $\mathfrak{U}_Z := \{\, U \cap (Z \times Z) \mid U \in \mathfrak{U} \,\} \in \tau_Z(B \cap Z)$ follows. Since by hypothesis f is b-continuous, we claim $(f \times f)(\mathfrak{U}_Z) \in \tau(f[B \cap \cap Z])$. Consequently we get $\overline{(f \times f)(\mathfrak{U}_Z)} \in f[B]$, since τ is isoform. But $\overline{(f \times f)(\mathfrak{U}_Z)} \subseteq (f^* \times f^*)(\mathfrak{U})$ implies $(f^* \times f^*)(\mathfrak{U}) \in \tau^*(f[B])$, which concludes the proof. $\Diamond$

4.9 Remark. As pointed out earlier, each preuniform convergence space (X, J_X) induces a corresponding b-convergence space (PX, τ_X) (see also Ex. 2.2(iii)).

Now let us call a b-convergence space $(\mathcal{B}^X, \tau)$ *saturated*, if
(Sat) $X \in \mathcal{B}^X$.
Conversely, given a saturated b-convergence space $(\mathcal{B}^X, \Gamma)$, we define the underlying preuniform convergence as follows:

$$\mathcal{L}_\Gamma := \Gamma(X)$$

Moreover, we note that the b-convergence τ_X is *equiform*, which means the following:
(e) $B, B' \in \mathcal{B}^X \setminus \{\emptyset\}$ implies $\Gamma(B) = \Gamma(B')$.

4.10 Definition. We call a saturated and equiform b-convergence space *preuniform.*

4.11 Theorem. *The category* ***PUCONV*** *with preuniform convergence spaces as objects and uniformly continuous maps between them* *as morphisms is isomorphic to the category* ***pub-CONV*** *of preuniform b-convergence spaces and b-continuous functions.*

Proof. This is immediately clear. $\Diamond$

4.12 Remark. Special cases of preuniform convergences are the so-called *principal* ones; i.e., a preuniform convergence space (X, J_X) is called a *principal preuniform convergence space*, provided there is a filter $\mathfrak{U}$ on $X \times X$ such that $J_X := [\mathfrak{U}]$, where

$$[\mathfrak{U}] := \{\mathfrak{V} \in \boldsymbol{FIL}(X \times X) \mid \mathfrak{V} \supseteq \mathfrak{U}\}.$$

Now it is easy to verify how diagonal filters, especially semiuniformities, quasiuniformities or uniformities, can be described by means of their corresponding preuniform b-convergences, namely as the *pointed* ones.

4.13 Definition. A b-convergence τ on $\mathcal{B}^X$ and the corresponding pair $(\mathcal{B}^X, \tau)$ are called *pointed*, provided

(p) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ implies $\tau(B) = \bigcup\{\, \tau(\{x\}) \mid x \in B\,\}$.

4.14 Remark. Hence, pointed b-convergences are of interest when studying spaces dealing with "uniform aspects". Moreover, we note that pointed b-convergence spaces are necessarily *isoform*.

The considerations above allow us to consider further specializations, for instance by setting $\mathcal{B}^X := \{\emptyset\} \cup \{\, \{x\} \mid x \in X\,\}$. So in case of having a b-convergence in $\mathcal{B}^X$, this leads us to a corresponding "generalized" convergence relation, and vice versa, so that the category ***GCONV*** of generalized convergence spaces and related maps can be considered as such an isomorphic one [9] with respect to Ex. 2.2(i), defining topological spaces in special cases. Moreover, we note that the corresponding b-convergence is necessarily pointed.

4.15 Theorem. *The category* ***pb-CONV****, whose objects are the pointed b-convergence spaces, is bicoreflective in* ***ib-CONV****.*

Proof. Given an isoform b-convergence space $(\mathcal{B}^X, \tau)$, for each $B \in \mathcal{B}^X$ we set

$$\tau^p(B) := \begin{cases} \{P(X \times X)\}, & \text{if } B = \emptyset; \\ \{\mathfrak{U} \in \boldsymbol{FIL}(X \times X) \mid \exists x \in B.\, \mathfrak{U} \in \tau(\{x\})\} & \text{otherwise.} \end{cases}$$

Hence $(\mathcal{B}^X, \tau^p)$ is a pointed b-convergence space. Evidently, the axioms (bC1) through (bC6) are satisfied. Now consider $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathfrak{U} \in$ $\in \tau^p(B)$. Then we have $\mathfrak{U} \in \tau(\{x\})$ for some $x \in B$, hence $\mathfrak{U} \in \tau^p(\{x\})$, which shows $\mathfrak{U} \in \bigcup\{\, \tau^p(\{x\}) \mid x \in B\,\}$.

Conversely, $\mathfrak{U} \in \bigcup\{\, \tau^p(\{x\}) \mid x \in B\,\}$ implies $\mathfrak{U} \in \tau^p(\{x'\})$ for some $x' \in B$. Therefore we obtain $\mathfrak{U} \in \tau(\{x\})$, which shows $\mathfrak{U} \in \tau^p(B)$.

Moreover, the identity map 1_X is b-continuous on $(\mathcal{B}^X, \tau^p)$, i.e., $\mathfrak{U} \in \tau^p(B)$, and without restriction $B \neq \emptyset$ implies $\mathfrak{U} \in \tau(\{x\})$ for some $x \in B$. Since τ is isoform, $\mathfrak{U} \in \tau(B)$ results, which concludes this verification.

Now let $(\mathcal{B}^Y, \Gamma)$ be a pointed b-convergence space and $f : (\mathcal{B}^Y, \Gamma) \to (\mathcal{B}^X, \tau)$ a b-continuous function. We have to show that $f : (\mathcal{B}^Y, \Gamma) \to (\mathcal{B}^X, \tau^p)$ is again b-continuous. Without restriction let $\emptyset \neq B \in \mathcal{B}^Y$. Then $\mathfrak{U} \in \Gamma(B)$ implies the existence of some $y \in B$ such that $\mathfrak{U} \in \Gamma(\{y\})$, since Γ is pointed. Hence $(f \times f)(\mathfrak{U}) \in \tau^p(\{f(y)\})$ by hypothesis. Consequently, we get $(f \times f)(\mathfrak{U}) \in \tau(\{f(y)\}) \subseteq \tau(f[B])$. $\Diamond$

4.16 Remark. Since ***pb-CONV*** is bicoreflective in ***ib-CONV*** (see Th. 4.15), it is again a topological category. Thus quotients and sums in ***pb-CONV*** are formed as in ***b-CONV***, whereas subspaces and products arise from the corresponding structures in ***ib-CONV*** by applying the corresponding bicoreflection.

4.17 Corollary. *For an isoform b-convergence space $(\mathcal{B}^X, \tau)$ the b-convergence τ^* is pointed iff τ is pointed.*

Proof. $B \in \mathcal{B}^* \setminus \{\emptyset\}$ implies $B \in \mathcal{B}^X$ or $B = \{\infty\}$. In the latter case $\tau^*(B) = \tau^*(\{\infty\}) = \{\dot{\infty} \times \dot{\infty}\} = \bigcup\{\, \tau^*(\{x\}) \mid x \in \{\infty\} = B\,\}$. In the first case consider $\mathfrak{U}^* \in \tau^*(B)$, hence there exists $\mathfrak{U} \in \tau(B)$ with $\mathfrak{U}^* \supseteq \bar{\mathfrak{U}}$. Because τ is pointed, $\mathfrak{U} \in \tau(\{x\})$ follows, which shows that $\mathfrak{U}^* \in \tau^*(\{x\})$. Conversely, let τ^* be pointed. $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathfrak{U} \in \tau(B)$ imply $\bar{\mathfrak{U}} \in \tau^*(B)$, hence by hypothesis $\bar{\mathfrak{U}} \in \tau^*(\{x\})$ follows. Now choose $\mathfrak{V} \in \tau(\{x\})$ with $\bar{\mathfrak{U}} \supseteq \mathfrak{V}$. Then $\mathfrak{V} \subseteq \mathfrak{U}$, because $V \in \mathfrak{V}$ implies $V = U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$ for some $U \in \mathfrak{U}$, which shows that $\mathfrak{U} \in \tau(\{x\})$.

4.18 Theorem. ***pb-CONV*** *is extensional.*

Proof. Use the results obtained above.

In the case of non-symmetric Convenient Topology, we will further check whether the category ***pb-CONV*** can serve as a topological universe.

4.19 Theorem. *For two pb-convergence spaces $(\mathcal{B}^X, \tau_X)$ and $(\mathcal{B}^Y, \tau_Y)$ consider the set $[\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}}$ of b-continuous functions $f : X \longrightarrow Y$ from $(\mathcal{B}^X, \tau_X)$ to $(\mathcal{B}^Y, \tau_Y)$. We define a b-convergence on the corresponding B-set $\mathcal{B}^{Y^X}$ (see also the beginning of Sec. 3) by setting for each $B^* \in \mathcal{B}^{Y^X} \setminus \{\emptyset\}$:*

$$\tau(B^*) := \big\{\mathfrak{U}^* \in \boldsymbol{FIL}([\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}} \times [\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}}) \mid \forall B \in \mathcal{B} \setminus \{\emptyset\}\ \forall \mathfrak{U} \in \tau_X(B).\ \mathfrak{U}^*(\mathfrak{U}) \in \tau_Y(B^*(B))\big\}$$

where $\mathfrak{U}^*(\mathfrak{U})$ *denotes the filter generated by the set* $\{\, U^*(U) \mid U^* \in \mathfrak{U}^* \wedge \wedge\, U \in \mathfrak{U} \,\}$, *with*

$$U^*(U) := \{\, (f_1(x_1), f_2(x_2)) \mid (f_1, f_2) \in U^* \wedge (x_1, x_2) \in U \,\}$$

and $B^*(B) := \big\{f(b) \mid f \in B^* \wedge b \in B\big\}$. *Further we set*

$$\tau(\emptyset) := \big\{P([\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}} \times [\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}})\big\}.$$

Then τ *is the* natural *function space structure on* $[\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}}$ *in* ***pb-CONV***.

Proof. By construction it only remains to prove the axioms (bc4), (bc6) and (pb), respectively.

To (bc4): We have $\dot f \times \dot f \in \tau(\{f\})$, since by hypothesis $\dot f \times \dot f(\mathfrak{U}) = = (f \times f)(\mathfrak{U}) \in \tau_Y(\{f\}(B)) = \tau_Y(f[B])$.

To (bc6): The inclusion $\mathfrak{U}^*(\mathfrak{U}) \subseteq \mathfrak{V}^*(\mathfrak{U})$ is valid for every $\mathfrak{U}^*, \mathfrak{V}^* \in \in \boldsymbol{FIL}([\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{b}} \times [\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{b}})$ with $\mathfrak{U}^* \subseteq \mathfrak{V}^*$ and each $\mathfrak{U} \in \tau_X(B)$ where $B \in \mathcal{B}^X$.

To (pb): For $\mathcal{B}^* \in \mathcal{B}^{Y^X} \setminus \{\emptyset\}$ consider $\mathfrak{U}^* \in \tau(B^*)$ and choose $f \in \in \mathcal{B}^*$. Then $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathfrak{U} \in \tau_X(B)$ imply $\mathfrak{U} \in \tau_X(\{x\})$ for some $x \in B$, since τ_X is pointed. By hypothesis $\mathfrak{U}^*(\mathfrak{U}) \in \tau_Y(\mathcal{B}^*(\{x\})$ follows, hence $\mathfrak{U}^*(\mathfrak{U}) \in \tau_Y(\{f(x)\}) = \tau_Y(\{f\}(\{x\}))$, because τ_Y is pointed as well, which yields $\mathfrak{U}^* \in \tau(\{f\})$.

The evaluation map

$$e : (\mathcal{B}^X, \tau_X) \times ([\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}}, \tau) \longrightarrow (\mathcal{B}^Y, \tau_Y)$$

is b-continuous, since the following equation

$$(e \times e)(\mathfrak{U} \times \mathfrak{U}^*) = \mathfrak{U}^*(\mathfrak{U})$$

holds for each $\mathfrak{U} \in \tau_X(B)$ and $\mathfrak{U}^* \in \tau(\mathcal{B}^*)$, where $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{B}^* \in \mathcal{B}^{Y^X} \setminus \{\emptyset\}$.

Moreover, finite products in ***pb-CONV*** can be described as follows:

$$(\tau_X \times \tau_Y)(B) := \big\{\mathfrak{U} \in \boldsymbol{FIL}(X \times X, Y \times Y) \mid \exists \mathfrak{U}_X \in \tau_X(p_X[B])\ \exists \mathfrak{U}_Y \in \tau_Y(p_Y[B]).\ \mathfrak{U} \supseteq \mathfrak{U}_X \times \mathfrak{U}_Y\big\}$$

where $B \neq \emptyset$ and p_X, p_Y denote the corresponding projections.

Now, in fact, let $\mathfrak{U} \in (\tau_X \times \tau)(B)$ for some $B \in \mathcal{B}^{X \times Y^X} \setminus \{\emptyset\}$ hence $\mathfrak{U} \in (\tau_X \times \tau)(\{x, f\})$ for some $(x, f) \in B$, since $\tau_X \times \tau$ is pointed.

$\mathfrak{U} \supseteq \mathfrak{U}_X \times \mathfrak{U}^*$ for some $\mathfrak{U}_X \in \tau_X(p_X[\{(x,f)\}])$ and $\mathfrak{U}^* \in \tau(p_{Y^X}[\{(x,f)\}])$, hence $(e \times e)(\mathfrak{U}_X \times \mathfrak{U}^*) = \mathfrak{U}^*(\mathfrak{U}_X) \in \tau_Y(\{f\}\{x\})$ by definition of τ.

Consequently, we obtain

$$\tau_Y(\{f\}\{x\}) = \tau_Y(\{f(x)\}) = \tau_Y(\{e(x,f)\}) \subseteq \tau_Y(e[B])$$

since τ_Y in particular is isoform.

Now, let $(\mathcal{B}^Z, \tau_Z)$ be a pointed b-convergence space and let $f : (\mathcal{B}^X, \tau_X) \times (\mathcal{B}^Z, \tau_Z) \longrightarrow (\mathcal{B}^Y, \tau_Y)$ be a b-continuous function. Then the map $\hat{f} : (\mathcal{B}^Z, \tau_Z) \longrightarrow ([\mathcal{B}^X, \mathcal{B}^Y]_{\mathrm{pb}}, \tau)$ defined by $\hat{f}(z)(x) := f(x,z)$ for every $x \in X$, $z \in Z$, is again b-continuous (see (2) in Sec. 3).

Suppose $\mathfrak{V} \in \tau_Z(\hat{B})$ for some $\hat{B} \in \mathcal{B}^Z \setminus \{\emptyset\}$. Then $\hat{B} \in \tau_Z(z)$ for some $z \in \hat{B}$. We aim to show that $(\hat{f} \times \hat{f})(\mathfrak{V}) \in \tau(\{\hat{f}(z)\})$ holds, because this implies $(\hat{f} \times \hat{f})(\mathfrak{V}) \in \tau(\hat{f}[\hat{B}])$, since τ in particular is isoform.

So let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathfrak{U} \in \tau_X(B)$, hence $\mathfrak{U} \in \tau_X(\{x\})$ for some $x \in B$. Consequently $(\hat{f} \times \hat{f})(\mathfrak{V})(\mathfrak{U}) = (f \times f)(\mathfrak{U} \times \mathfrak{V}) \in \tau_Y(f[\{(x,z)\}])$ follows, which shows that $(\hat{f} \times \hat{f})(\mathfrak{V}) \in \tau(\{\hat{f}(z)\}(\{x\})$ holds, which implies $(\hat{f} \times \hat{f})(\mathfrak{V}) \in \tau(\{\hat{f}(z)\})$. This concludes the proof. $\Diamond$

4.20 Corollary. ***pb-CONV*** *is cartesian closed.*

4.21 Theorem. ***pb-CONV*** *is a topological universe.*

Proof. Taking into account Rem. 4.16, Th. 4.19 and Th. 4.18, the claim easily follows.

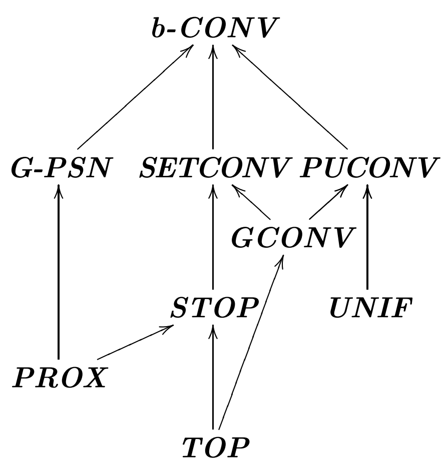

Figure 1. Relationships of the categories mentioned in this paper

4.22 Remark. Since ***pb-CONV*** is bicoreflective in ***ib-CONV*** (see Th. 4.15) and closed under formation of products in ***ib-CONV***, ***pb-CONV*** is again strong.

4.23 Theorem. ***pb-CONV*** *is a strong topological universe.*

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